Infinitesimals and Infinites in the History of Mathematics: A Brief Survey*

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Abstract: We will review the main episodes of the history of the infinite in mathematics, stressing the asymmetry between infinitesimals and infinites. While the actual infinite, after being avoided for so long, when it became a mathematical object it entered mathematics to stay, infinitesimals have had a troubled history of condemnations and resurrections. This has probably to do with the ambiguous basic act of mathematization as revealed by the passage from the Egyptians' ropes to the length without largeness of Euclid's definition of the line. The main characters of the narration are Aristotle, Archimedes, Cavalieri, Euler, Cantor. In the end we will give some information on the last re-emergence of infinitesimals, in nonstandard analysis, and on some quite recent and intriguing new ideas.

1 Introduction

The history of infinitesimals and infinities in mathematics present a curious asymmetry, that invites speculation. The actual infinite was not considered in mathematics for thousands of years, but when it entered, it came to stay. Infinitesimals have had a troubled life of emergence and submergence; mathematicians were always attracted to infinitesimals but unable to give precise rules to handle them. We suspect that this difficulty has to do with a psychological uneasiness about the basic act of mathematization: to transform the ropes of Egyptians in a mathematical object one has to subtract thickness, to imagine them thinner and thinner untill their largeness vanishes, although they do not; apparently the ropes suddenly enter another realm, as Alice through the looking glass. Euclid's definition was that a line is a length

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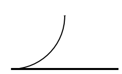
without largeness (*Elements*, I, Def. 2.). However when a figure is filled up with parallel segments, as it happens in applications to measure, lines seem to have a largeness which at the same time is and is not zero, since they fill an area. Similarly for “point”, which has no parts: at the beginning the Greeks used the word for the point of an arrow, $\sigma\tau\iota\gamma\mu\acute{\eta}$, then with Euclid the immaterial “sign” (which is not a material “mark”). These first stipulations are the source of the many paradoxes of the infinite throughout history.

We will briefly survey a few of the more significant episodes of the infinities saga, without trying to answer this conundrum; at long last, we seem to have learned how to treat this concept both in the large and in the small, but only thanks to the fact that we have changed the way we do mathematics.

2 Aristotle’s and the potential infinite

The Greeks were well aware of the possibility of infinitesimal quantities, e.g. the so called “horn”, or the angle of contingency:



There are documents of debates on how to measure such an angle (see [16], vol. 2, pp. 39-43).

Antiphon of Athens, a sophist living in Socrates’ time, tried to square the circle by a method that anticipated the exhaustion method, namely by inscribing a polygon with sides so small that the polygon would be undistinguishable from the circle.

But the standard was fixed by Euclid, and it left out infinitesimals:

> Two quantities are said to have a ratio, the one to the other, when, if multiplied, they can override themselves (Euclid, V, Def. 4).

To restrict oneself to quantities having a ratio the one to the other amounted to assume what was later called Archimedes’ principle (“given two quantities $a < b$ there exists an n such that $n \cdot a \geq b$”)[1] and to negate infinitesimals: if a is infinitesimal and b is finite, $n \cdot a < b$ for all n.

[1]Archimedes himself called it “Eudoxos’ Lemma”, in a form reverse to that by Euclid in *Elements*, X 1: “Given two different quantities, if from the greater one subtracts a quantity greater than its own half, and from the rest again a quantity greater than its

While avoiding infinitesimals, the Greeks also eschewed the actual infinite. Their practice was sanctioned by Aristotle (384-322): the infinite is the potential infinite (*apeiron*, unlimited).

> Infinite [is said in general of] that from which it is always possible to subtract something, and what is subtracted, besides being finite, is always different ([2], III, 6, 206 *a* 27).

A quantity according to Aristotle can be infinite by division or by addiction, where a quantity a is infinite by addiction if it can be obtained as, in modern notation,

$$a = \frac{1}{2}a + \frac{1}{2^2}a + \frac{1}{2^3}a + \ldots,$$

while a quantity is infinite by division if it is divisible in parts infinite by division. Continuous quantities are infinite by division; a line is not composed of points nor of other indivisible parts. A point according to Aristotle is the extreme of a segment, or what you get when you cut a segment.

The *apeiron* cannot be present in its totality to the thought, because it is not that outside which there is nothing, but it is that "outside which there is always something" ([2], 207 a 1).

So the mathematicians did not assume the infinite as an object of thought, but confined themselves to increase the finite, according to Euclid's second postulate: "To extend a finite segment continuously in a segment".

> This [Aristotle's] argument [against the actual infinite] does not destroy the investigations of the mathematicians, with the negation of the existence of the infinite [...] because now they do not need the infinite and make no use of it, but only require, postulate, that the finite segment can be lengthened as much as they want [...] ([2], 207 *b* 27-33).

However, in the exhaustion method of Eudoxos (408-355), a sort of process to the limit conceived to avoid infinitesimals, the potential infinite was too dangerously close to the actual infinite. And besides orthodoxy, there were more direct methods which were inspired by the infinitesimals. The best example is given by Archimedes (287-212).

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half, and this operation is repeated continuously, one will obtain a quantity smaller than the smaller of the given ones".

3 Archimedes' infinitesimals

In his *To Erathostenes: Method on mechanical theorems* Archimedes presents us with arguments that are not flawless undisputable proofs, but have a strong heuristic power:

> [...] our arguments are not true demonstrations, but it is easier to find one when one becomes conversant with the subject thanks to the informations gathered with the method [...]

Then in the second part of his work, he gave also the more acceptable geometric proofs. There were actually two flaws in his arguments: one had to do with the use of the physical concept of the lever, the other with infinitesimals. The first one could have been simply disposed with the mathematical treatment of statics, due also to Archimedes, not so the second.

Let us recall the example of the area of the parabolic segment, based on this famous figure:

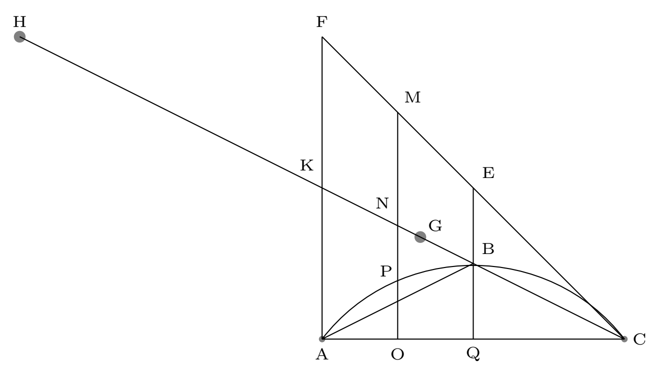

Figure 1: Archimedes' mechanical proof.

He knows the following properties of the parabola

$$EB = BQ, FK = KA, MN = NO$$

and

$$CA : AO = MO : OP$$

$$CA : AO = CK : KN$$

whence

$$HK : KN = MO : OP.$$

Consider K as the pivot of a balance with beams HK and KC. Consider a segment $TS = OP$ centered in H. It balances MO in the sense that

$$HK : KN = MO : TS,$$

Now the segments OP with O varying in AC fill in the parabolic segment ABC, *segm.*ABC, while the segments MO fill in the triangle CFA. The point G on CK such that

$$CK = 3KG$$

is the center of mass of CFA so that

$$CFA : segm.ABC = HK : KG$$

hence

$$segm.ABC = 4/3ABC. \quad \square$$

The proof that Archimedes calls geometric, meaning that it is done with Euclid's tools, requires the exhaustion method.

Given the triangle ABC inscribed in the parabolic segment, one inscribes the two triangles AEB and BFC, and from properties of the parabola one has

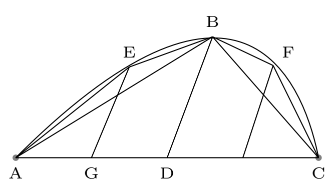

Figure 2: The geometric proof.

$$BD = \frac{4}{3}EG$$

and

$$ABC = 8 \cdot AEB = 8 \cdot BFC$$

$$AEB + BFC = \tfrac{1}{4}ABC.$$

If, in the same way, one inscribes a triangle in each of the segments AE, EB, BF, FC, each of them will be 1/8 of AEB or BFC, their total area 1/16 of ABC.

By iteration one will get an inscribed polygon of area

$$S_n = (1 + \frac{1}{4} + \frac{1}{4^2} + \ldots + \frac{1}{4^{n-1}})ABC$$

and obviously $S_n < P$, P being the area of the parabolic segment.

Archimedes proves also that

$$S_n = (\frac{4}{3} - \frac{1}{3}\frac{1}{4^{n-1}})(ABC)$$

hence in particular

$$S_n < \frac{4}{3}(ABC).$$

Now the exhaustion trick is invoked. We would conclude that $\lim S_n = P = \frac{4}{3}(ABC)$, but the Greeks followed an indirect route. They showed that $P \leq \frac{4}{3}(ABC)$ and that $P \geq \frac{4}{3}(ABC)$, using the indefinite approximation of P by the S_n's and reasoning by absurd:

1. It is impossible that $P > \frac{4}{3}(ABC)$.
   If $P > \frac{4}{3}(ABC)$, there should be an n such that

   $$S_n > \frac{4}{3}(ABC)$$

   which is impossible, since in this case the construction of the S_n's could go on in the remaining area.

2. It is impossible that $P < \frac{4}{3}(ABC)$,
   since in this case for an n such that

   $$\frac{1}{3}\frac{1}{4^{n-1}}(ABC) < \frac{4}{3}(ABC) - P$$

   one would have

   $$\frac{4}{3}(ABC) = S_n + \frac{1}{3}\frac{1}{4^{n-1}}(ABC)$$

   $$\frac{4}{3}(ABC) < S_n + \frac{4}{3}(ABC) - P$$

   $$S_n > P,$$

   which is impossible.

Hence $P = \frac{4}{3}ABC$. □

4 Cavalieri's indivisibles

Infinitesimals' second life unrolls in the seventeenth century, thanks mainly to Bonaventura Cavalieri (1598-1647), besides other preceding authors such as Nicolas of Oresme (see [6], p. 81). Cavalieri was the "new Archimedes", as Galileo and his contemporaries called him. For Cavalieri, figures are made of the totality of their sections, in particular plane figures are made of the totality of the segments intercepted by a perpendicular plane moving parallel to itself. He called this totality "all the lines of the figure" (*omnes lineae figurae*). Similarly for three dimensional solids.

He calculated areas and volumes of figures and solids, notably the areas under polynomial curves given later by the integrals

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1},$$

using the following principle:

> Two plane figures have the same ratio one to the other as all their [straight] lines taken in an arbitrary direction.

More explicitely: "plane figures enclosed by the same parallel lines, and such that any line parallel to that same direction cuts the figures in segments having always the same ratio have one to the other the same ratio as that of the intersections" (*Geometria*, Liber VII, quoted by [12], p. 32.). His *motto* was: "ut unum ad unum, sic omnia ad omnia" (as one to one, so all [lines] to all).

Cavalieri did not take position on the physical and metaphysical problems of the composition of the continuum, and took care to avoid equating it with the mathematical frame of the infinitesimals. His attitude was shared by many of his contemporaries, and neatly summed up by one of them:

> Indivisible quantities are considered in different manner by philosophers and by mathematicians: the first call indivisibles the minimal and ultimate parts of the continuum, which constitute it and give it extension, although they in themselves have no extension and cannot be further divided in smaller parts; the latter accept

them independently by any matter, and that have, or may have a ratio of measure, in comparison with other quantities.[2]

Galileo Galilei (1564-1642) considered indivisible as a philosopher, and was convinced that the indivisibles of two figures could not be compared to each other; in any case they were infinite, and no relation of less or greater could be defined, since all infinites are the same (he relied, among others, on the fact that the infinite of the even numbers was the same of that of all natural numbers).

In some of his proofs, however, Galileo had to resort to the same arguments as Cavalieri. For example in the discussion of the falling bodies Theorem 1 asserts that the time in which a body starting from rest and moving of uniformly accelerated motion covers a given distance is the same of that of the same body moving with a uniform velocity equal to the one half of the final velocity in the uniformly accelerated motion.

For the proof, in [11], p. 776, he used the diagram shown in Fig. 3.

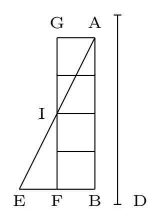

Figure 3: Galileo's indivisibles

"All the lines" parallel to BE drawn from points of AB, which are the velocities of the body moving in uniformly accelerated motion end on AE, and fill the same figure as "all the lines" from AB to GF which represent the constant velocity of a body moving in uniform motion.

In analogous way he proved in *Teorema II, Propositione II* that the distance is proportional to the square of the time, given that the distances grows as the progression of the odd numbers:

[2] "Indivisibilia in Quantitate, aliter considerantur a Philosophis, aliter a Mathematicis: illi, prout appellant minimas & ultimas partes continui collectim illud integrantes, & facientes extensum; quae tamen in seipsis sint inextensa, nec ulterius in alias inferiores minutias dividi possint; Isti, praescindendo ab omne materia, accipiunt, prout habent, vel habere possunt rationem mensurae, saltem comparata ad alias quantitates", H. Vitali, *Lexicon Mathematicum*, 1690, quoted by [12], p. 41.

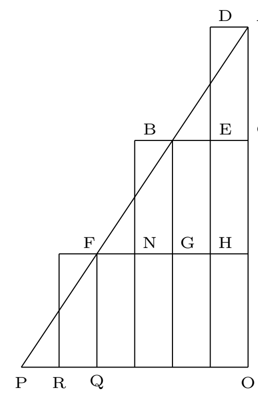

Figure 4: Teorema II, Propositione II

This was the geometrical way to integration used before the infinitesimal calculus.

5 The infinitesimal calculus

Geometrical infinitesimals were done away by the definition of the derivative and the differential and integral calculus, but they still lingered as variables quantities. Newton's fluxions and fluents, and his definitions appealing to "vanishing increments" and "ultimate ratio" were too reminiscent of infinitesimals, besides appearing contradictory. Berkeley's criticism in his 1734 *The Analyst* was merciless.

Gottfried W. Leibniz (1646-1716) on the Continent was able to set up a calculus which avoided metaphysical considerations, though his dx and dy are infinitesimals. Leibniz struggled his whole life to justify, first of all to himself, the infinitesimal methods. He formulated a kind of continuity principle;

> We conceive the infinitely small not as a simple absolute zero, but as a relative zero [...] that is as a vanishing quantity which however maintains the character of what is vanishing ([19], vol. 4, p. 218).
>
> In any supposed transition, which ends up in a final result, it is admissible to develop a general argument [concerning the transition] such that it comprises also the final result ([19], vol. 3, p. 524).

Among several oscillations, he fundamentally kept to the following principles: that infinitesimals and infinities do not have a metaphysical reality; that they are useful fictions to facilitate discovery, but they can be shunned by approximations, as the ancients did; that they are to be considered ideal elements, as the imaginary numbers, which satisfy the same algebraic laws of ordinary numbers.

> On n'as pas besoin de prendre l'infini ici à la rigueur, mais seulement comme lorsqu'on dit dans l'optique, que les rayons du soleil viennent d'un point infiniment éloigné et ainsi son estimés parallèles. Et quant il y a plusieur degrés d'infini ou infiniment petits, c'est comme le globe de la terre est estimé un point à l'égard de la distance des fixes, et une boule que nous manions est ancor un point en comparaison du semidiamètre du globe de la terre, de sort que la distance des fixes est un infiniment infini ou infini de l'infini par rapport au diamètre de la boule. Car au lieu de l'infini ou de l'infiniment petit, on prend des quantités aussi grandes et aussi petites qu'il faut pour que l'erreur soit moindre que l'erreur donnée, de sorte qu'on ne diffère du style d'Archimède que dans l'expressions, qui sont plus directes dans notre méthode et plus conformes à l'art d'inventer [...]
>
> D'où il s'ensuit, que si quelqu'un n'admet point des lignes infinies et infiniment petites à la rigueur métaphysique et comme des choses réelles, il peut s'en servir sûrement comme des notions idéals qui abrègent le raisonnement, semblable à ce qu'on appelle racines imaginaires dans l'analyse commune (comme par exemple $\sqrt{-2}$) [...]
>
> [...] et il se trouve que les règles du fini réussissent dans l'infini comme s'il y avait des atomes (c'est à dire des éléments assignables de la nature) quoiqu'il n'y en ait point la matière étant actuellement sousdivisée sans fin; et que vice versa les règles de l'infini réussissent dans le fini, comme s'il y'avait des infiniment petits métaphysiques, quoiqu'on n'en n'ait point besoin; et que la division de la matière ne parvienne jamais à des parcelles infiniment petites: c'est pars que tout se gouverne par raison, et qu'autrement il n'aurait point de science ni règle, ce qui ne serait point conforme avec la nature du souverain principe ([18], p. 350).

It was not clear, however, which were the laws holding equally well for the finite and the infinite. Archimedes' principle

$$\forall a, b\,(a > b > 0 \to \exists n \in \mathbb{N}(nb > a))$$

cannot hold if one allows infinitesimal quantities.

6 Euler

It has been Leonhard Euler (1707-1783) who exploited at the utmost in a free and easy way the leibnizian ambiguity of the identity of algebraic laws. With Euler, infinitesimals reach the highest peak of success, together with infinite numbers. For Euler, ∞ was not the last of the numbers, as was sometimes assumed in notations such as $\lim_\infty$ or $\sum_1^\infty$; since it was a legitimate number, one could apply to it the arithmetical operations, thus obtaining with each infinite n also $n-1$ and $n+1$ and so on.

Infinitesimals also were numbers, namely the inverse of the infinite numbers, but being less than any assignable quantity they had to be zero; however they were denoted by the symbol ω different from "0". Euler artfully used interchangeably x and $x+\omega$; as for infinitesimals of higher order he simply put equal to 0 earlier than those of lesser order.

In his computations, Euler equated a series with an infinite polynomial, thus making use of the product decomposition of the latter. His first success with this technique was the solution of the Basel problem, posed by Jacob Bernoulli, of finding the value

$$\pi^2/6 = 1 + \frac{1}{4} + \frac{1}{9} + \ldots + \frac{1}{n^2} + \ldots$$

However Euler continued to search, as Archimedes did, for a proof which would be acceptable according to the standards of his time, until he found one.[3]

As an example of Euler's approach, let us recall his discovery of the series for $\cos x$ and $\sin x$.

Given De Moivre's formulae

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n$$

[3]See Polya in [21], p. 21. In his proof Euler used integrals, see [9], pp. 55-7.

and

$$\cos n\theta - i\sin n\theta = (\cos\theta - i\sin\theta)^n$$

apply the binomial formula, after summing and dividing by 2, to obtain

$$\begin{aligned}\cos n\theta &= \tfrac{1}{2}\left[\cos^n\theta + \tfrac{ni\cos^{n-1}\theta\sin\theta}{1} - \tfrac{n(n-1)\cos^{n-2}\theta\sin^2\theta}{2!} + \dots\right] \\ &\quad + \tfrac{1}{2}\left[\cos^n\theta - \tfrac{ni\cos^{n-1}\theta\sin\theta}{1} - \tfrac{n(n-1)\cos^{n-2}\theta\sin^2\theta}{2!} + \dots\right] \\ &= \cos^n\theta - \tfrac{n(n-1)\cos^{n-2}\theta\sin^2\theta}{2!} + \tfrac{n(n-1)(n-2)(n-3)\cos^{n-4}\theta\sin^4\theta}{4!} - \dots\end{aligned}$$

If you put $x = n\theta$ with n infinite and θ infinitesimal, $\cos\theta$ becomes 1 and $\sin\theta$ becomes θ; $n, n-1, \dots$ can be identified (for the old principle of the annihilation of the number[4]) and you get

$$\begin{aligned}\cos x &= 1^n - \tfrac{n\cdot n\cdot(1)^{n-2}(x/n)^2}{2!} + \tfrac{n\cdot n\cdot n\cdot n\cdot(1)^{n-4}(x/n)^4}{4!} - \dots \\ &= 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \dots\end{aligned}$$

Similarly for $\sin x$. Even the most celebrated formula of Euler, and of the whole mathematics,

$$e^{ix} = (\cos x + i\sin x)$$

is obtained in the same way, starting from

$$\begin{aligned}\cos x = \cos n\theta &= \tfrac{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n}{2} \\ &= \tfrac{(1+ix/n)^n + (1-ix/n)^n}{2}.\end{aligned}$$

Sometimes even Euler was not above his time; he said for instance that ∞ is negative, since $1/\infty$ is less than every positive number; and he gave strange values (though always with some clever motivation) to divergent series.

[4]Thus was called in the Middle Ages Aristotle's remark that a plus ∞ would give ∞.

7 The triumph of the infinite, and the ban on infinitesimals

With the ε, δ definition of limit, infinitesimals were expelled from calculus; at the same time, while Weierstrass' revolution was not yet completely assimilated, Richard Dedekind (1831-1916) and Georg Cantor (1845-1918) introduced the actual infinite in mathematics. They began with the definition of the real numbers, which was necessary as the appropriate setting for the notion of limit, then Cantor went on to abstract sets.

This origins of set theory in the analysis of nineteenth century explains why the legitimation of the infinite numbers (although different from those of Euler) did not go on a par with a similar legitimation of the infinitesimals. On the contrary, Cantor was one of their fiercest enemy.

His arguments were put forward especially in the debate with Giuseppe Veronese (1854-1917), author of a book dedicated to geometry in many dimensions and with many sorts of linear unities, [28], where he advocated also the use of non-Archimedean fields. Cantor's arguments were weak and unsubstantiated: he claimed for instance that if ω infinitesimal, $\omega \cdot \theta$ infinitesimal even for θ infinite. The psychological reason of his opposition was probably that he was sure of the completeness of the system of the reals, as he had defined them, which excluded any enlargement. But when he tried on the basis of this intuition to refute the possible existence of infinitesimals, his purported proof was circular since it relied on the truth of Archimedes' axiom. This axiom is a consequence of the completeness of the real numbers system as formulated axiomatically by David Hilbert in 1900.

When Veronese tried to appease him by saying "I start from other hypotheses", Cantor answered with the inscription of the Newtonian quotation of "Hypotheses non fingo", prefixed to his last work "Beiträge zur Begründung der transfiniten Mengenlehre" ([7]).

He had not been influenced by the new trend of axiomatic mathematics; for him, to do mathematics meant to give the right definitions, and proceed henceforth by pure reason.

8 The return of the infinitesimals

A foundations of infinitesimals which is unobjectionable from the point of view of the rigor has been given in the nineteenth century by Abraham

Robinson (1918-1974), in 1965, see [22]. His solution does not only allow a treatment of analysis (and topology, measure theory, actually every mathematical discipline) on the basis of non-Archimedean fields: it realizes the Leibniz's request that the ideal elements satisfy the same laws of the ordinary ones.

The solution of the riddle has been possible thanks to a few concepts and results due to the mathematical logic of the twentieth century: first of all the notion of language, so that the laws one wants to preserve can be specified as those expressed in the original first order language of the classical structure to be enlarged.

The old and the enlarged structure are linked by a transfer principle, according to which the same statements expressible in the fixed language hold in both structures. If one starts from a model of the real numbers, in the enlargement the Archimedes' axiom does not hold because the natural numbers are not definable in the reals, and quantifiers cannot be constrained to vary on the original natural numbers. However it holds if the quantifier $\exists n$ varies on the new natural numbers, among which there are also the infinite ones (every infinite set has a natural extension in the enlargement).

The construction is possible for a theorem due to Kurt Gödel (1929), the completeness theorem for logic ([13]). In informal terms, it states:

> It is sufficient that the combinatorial condition of the non existence of a derivation of a contradiction from the specification of a concept holds, in order that we are able to conceive and imagine, and define, a concrete realization of the concept.

If the axioms are consistent, there exists a structure that is a model of the axioms. Consider a theory T, e. g. Peano's axioms for arithmetic, or the set of all the sentence true in $\mathbb{N}$; if T is consistent, as we may safely assume, also

$$\mathsf{T} \cup \{c \neq \underline{n} : n \in \mathbb{N}\}$$

is consistent, where c is a new symbol non occurring in T, and $\underline{n}$ a term of the language representing n.

If ${}^*\mathbb{N}$ is a model of T, its structure is something of this kind:

with a standard part followed by a non standard part made up of blocks, $\ldots a-n, \ldots a-1, a, a+1, \ldots, a+n, \ldots$, each of the order type of $\mathbb{Z}$.

But the picture is somewhat misleading, since the ordered structure of the non standard part of a model T is still more complicated: the set of copies of $\mathbb{Z}$ to the right of the picture must be an infinite dense set, for example of the order type of $\mathbb{Q}$ or $\mathbb{R}$.

By use of the transfer principle, one can easily prove for example

$$\sum_1^\infty \frac{1}{k(k+1)} = 1$$

à la Euler, in the following way: for infinite ω

$$\sum_1^\omega \frac{1}{k(k+1)} = \sum_1^\omega \frac{1}{k} - \sum_1^\omega \frac{1}{k+1} = 1 + (\sum_2^\omega \frac{1}{k}) - (\sum_2^\omega \frac{1}{k}) - \frac{1}{\omega+1} \approx 1$$

where $\approx$ means that the difference is infinitesimal. □

The conceptual apparatus of non standard analysis appears to be a vindication of Leibniz, since it allows us to accept as correct the old formulations at the origin of the infinitesimal calculus. For example the following theorem is proved:

Theorem A succession of standard reals $\{a_n\}_{n\in\mathbb{N}}$ converges to the standard limit l, for $n \to \infty$, if and only if a_ω is infinitely close to l for all infinite ω. □

The theorem gives a complete equivalence between the ϵ, δ definition of the limit and its characterization by means of infinitesimals.

9 New ideas

Now that we are confident that infinite natural numbers and their infinitesimal inverses can be treated together in a consistent way, we expect new experiments to find their way, in the true Cantorian (not Cantor's) spirit of "free mathematics". We will mention here the timely proposal of an enlarged numerical system advanced recently by Yaroslav D. Sergeyev.[5] This is simpler than non standard enlargements in its conception, it does not require infinitistic constructions and affords easier and stronger computation power.

Sergeyev does not need the apparatus of the axiomatic mathematics; he is a representative of the new philosophy of mathematics which assumes a stronger similarity between mathematics and physics. He relies in an essential

[5]See [24]; for more up to date presentations see [25] and Sergeyev's Lagrange Lecture [27].

way on the fact that only a finite number of operations, material or formal, can be executed for any task, and this is in itself a guarantee of consistency.[6]

Sergeyev postulates that the number of elements of $\mathbb{N}$ is a number he calls "infinite unity" and denotes by ①, the "grossone".

Every finite natural number n is such that $n <$ ①.

The usual algebraic laws hold:

$$0 \cdot ① = ① \cdot 0 = 0, ① - ① = 0, \frac{①}{①} = 1, ①^0 = 1, 1^{①} = 1,$$

and for all n, there exists $\frac{①}{n}$.

This last law follows from the fact that $\frac{①}{n}$ is the number of elements of the periodic sets $\mathbb{N}_{k,n} = \{k, k+n, k+2n, \ldots\}$.

In fact, Sergeyev's leading criterion is the Greeks' philosophical principle that "the part is less than the whole", one of Euclid's common notions.[7] He takes Euclid at his word, for all sets of natural numbers, not only for finite ones. So the definition of cardinality cannot be anymore the Cantorian one of the one-to-one correspondence. If ① is the number of elements of $\mathbb{N}$, $\frac{①}{2}$ is the number of elements of sets which have half of elements as $\mathbb{N}$, such as the set of even numbers. The same for the $\mathbb{N}_{k,n}$. Every proper subset of $\mathbb{N}$ has a different cardinality.

The set of natural numbers usually denoted by

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

[6]In an otherwise unaccountably bilious and spiteful review of Sergeyev's work, [15], Gutman and Kutateladze have actually done him a good service by pointing out that if you take a Robinson infinite natural number θ, the factorial $\theta!$ satisfies all the algebraic properties of ① (see below). This assures those who need reassurance that Sergeyev's system is as consistent as classical mathematics. Such a $\theta!$ however has none of the cardinal and ordinal properties of ① discussed below. For a different kind of review, see [23].

[7]For another approach inspired by the same principle see [3].

A different proposal is that of [17], where Lakoff and Nuñez consider a system of numbers obtained by adding just one infinite H and one infinitesimal $\delta = 1/\mathrm{H}$, which they call the system of granular numbers. The metaphor is that infinitesimals are the numbers we want to zoom in and look at with finer grain. δ is conceived as the termination of the process $n \mapsto 1/n$. They confine themselves to the treatment of limits, with no concern for the cardinal meaning of the new number. Sergeyev's intuition of infinite numbers as cardinalities of sets refutes Lakoff and Nuñez's general assumption that the infinite is always conceived as the termination of a process.

is now conceived as

$$\mathbb{N} = \{1, 2, 3, \ldots, ① - 2, ① - 1, ①\}.$$

The new set of natural numbers can be obviously extended by applying to ① the arithmetical operations, getting $① + 1, \ldots, ①^n, \ldots, 2^{①}, \ldots ①^{①}, \ldots$

Being a natural number, ① is simultaneously both a cardinal and an ordinal, without the vagaries of cantorian ordinal arithmetic: $n + ①$ is equal to $① + n$ as $n \cdot ①$ is equal to $① \cdot n$. Notice that since $2^{①} \neq ①^{①}$, finer distinctions are possibile in the nondenumerable domains.

Since every natural number can be used as a basis for the positional representation of numbers, the same function can be taken on by ①, using the extended system, so that every rational number can be written in the form

$$c_m ①^m + c_{m-1} ①^{m-1} + \ldots + c_1 ① + c_0 + c_{-1} ①^{-1} + c_{-2} ①^{-2} + \ldots + c_{-k} ①^{-k},$$

where the c_i's can be chosen, with suitable transformations, as finite numbers.

In this way the algebraic calculus extends immediately to infinite numbers and is easily programmable. It has actually been programmed on a dedicated computer.

A typical application is the solution of systems of linear equations, when pivots are null; 0 is substituted by $\frac{1}{①}$ and at the end of the computations one keeps only the finite part, discarding the infinitesimals, as Euler did.[8]

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