Infinigons of the hyperbolic plane and grossone

Maurice Margenstern
professor emeritus

Université de Lorraine,
Laboratoire d'Informatique Théorique et Appliquée, EA 3097,
Campus de Metz,
Île du Saulcy,
57045 Metz Cedex, France,
email margenstern@gmail.com
`http://www.lita.sciences.univ-metz.fr/ margens`

Abstract

In this paper, we study the contribution of the theory of grossone to the study of infinigons in the hyperbolic plane. We can see that the theory of grossone can help us to obtain a much more precise classification for these objects than in the traditional setting.

Keywords: tilings, hyperbolic plane, infinigons, grossone

In [4], an algorithmic approach to the infinigons was given by this author.

Infinigons of the hyperbolic plane are polygons with infinitely many sides. It is the case that there are infinitely many such objects and that, among them, there is an infinite family which tiles the hyperbolic plane by applying to an initial infinigon the process which is used to obtain a tessellation from an ordinary regular convex polygon of that plane. The existence of infinigons which tiles the plane appear already in [1] and in [10]. In [4], it was proved that for any angle α with $\alpha \in]0..\pi[$ it is possible to construct an infinigon such that consecutive sides make an angle of α. Moreover, such an infinigon tiles the plane by reflection in its sides and, recursively, of its images in their sides, when $\alpha = \dfrac{2\pi}{k}$ with k being a positive integer with $k \geq 3$ and only in this case. As already mentioned, [4] gives an algorithmic construction for the tiling defined by an infinigon whose angle is $\dfrac{2\pi}{k}$ with $k \geq 3$.

Of course, when we speak of an infinite object or infinitely many objects in the framework of grossone, it comes to our mind that we have to make use of more precise terms. When we speak of an infinite family of infinigons, we have of course to make more precise how infinite our family is and how infinite our infinigons themselves are.

In In Section 1, we remind the basic features of the numeral system we use for the results of the paper. In Section 2, we remind the algorithmic approach of [4] and then we revisit the classical definition. From this, we shall infer the new approach explained in Section 3.

1. Infinities and infinitesimals expressed in grossone-based numerals

Let us draw the attention of the reader of the study published in *Science*, see [2], where an Amazonian primitive tribe, Pirahã, is described. People of the tribe use a very simple numeral system[1] for counting: one, two, many. For them, all quantities larger than two are just 'many' and such operations as 2+2 and 2+1 give the same result, i.e., 'many'. Using their weak numeral system Pirahã are not able to see numbers 3, 4, etc., to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that. Moreover, the weakness of their numeral system leads to such results as

$$\text{`many'} + 1 = \text{`many'}, \qquad \text{`many'} + 2 = \text{`many'},$$

which are very familiar to us in the context of views on infinity used in the traditional calculus

$$\infty + 1 = \infty, \qquad \infty + 2 = \infty.$$

This analogy suggests that our difficulty in working with infinity is not connected to the nature of infinity. The difficulty is just a result of inadequate numeral systems used to express numbers. In fact, numeral systems strongly influence our capabilities to describe physical and mathematical objects. For instance, Roman numeral system has no numeral to express 0. As a consequence, the expression III-X in this numeral system is an indeterminate form. Moreover, any assertion regarding negative numbers and zero cannot be formulated using Roman numerals because there are no symbols corresponding to these concepts in this concrete numeral system.

The numeral system proposed in [13, 14, 18] is based on an infinite unit of measure expressed by the numeral ① called *grossone* and introduced as the number of elements of the set $\mathbb{N}$ of natural numbers. This is a clear difference with non-standard analysis in which non-standard infinite numbers are not connected to concrete infinite sets and do not belong to $\mathbb{N}$. Other symbols dealing with infinities and infinitesimals as ∞, Cantor's ω, $\aleph_0, \aleph_1, \ldots$, etc are not used together with ①. Similarly, when the positional numeral system and the numeral 0 expressing zero had been introduced, symbols V, X, and other symbols from the Roman numeral system had not been involved.

[1] We remind that *numeral* is a symbol or a group of symbols which represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols '3', 'three', and 'III' are different numerals, but they all represent the same number.

Note that people very often do not pay a great attention to the distinction between numbers and numerals. In this occasion it is necessary to recall constructivists who studied this issue. Many theories dealing with infinite and infinitesimal quantities have a symbolic character, not a numerical one. For instance, many versions of non-standard analysis are symbolic, since they have no numeral systems to express their numbers by a finite number of symbols. The finiteness of the number of symbols is necessary for organizing numerical computations. Namely, if we consider a finite n, then it can be taken as $n = 7$, or $n = 108$ or any other numeral used to express finite quantities and consisting of a finite number of symbols. In contrast, if we consider a non-standard infinite m then it is not clear which numerals can be used to assign a concrete value to m. One of the important differences between the new approach and non-standard analysis consists of the fact that the new numeral system allows us to assign concrete values to infinities and infinitesimals as it happens with finite values. In fact, we can assign $m = ①$, $m = 3① - 2$ or to use any other infinite numeral involving grossone to give a numerical value to m. See [13, 14, 18] for a detailed discussion.

The numeral ① allows one to construct different numerals expressing different infinities and infinitesimals and to execute numerical computations with all of them. As a result, in occasions requiring infinities and infinitesimals indeterminate forms and various kind of divergence are not present when one works with any finite, infinite, or infinitesimal numbers expressible in the new numeral system and it becomes possible to execute arithmetical operations with a variety of different infinities and infinitesimals. For example, taking ① and $①^{3.1}$ as examples of infinities and $①^{-1}$ with $①^{-3.1}$ as examples of infinitesimals, it follows that

$$0 \cdot ① = ① \cdot 0 = 0, \quad ① - ① = 0, \quad \frac{①}{①} = 1, \quad ①^0 = 1, \quad 1^{①} = 1, \quad 0^{①} = 0, \qquad (1)$$

$$0 \cdot ①^{-1} = ①^{-1} \cdot 0 = 0, \qquad ①^{3.1} > ①^1 > 1 > ①^{-1} > ①^{-3.1} > 0,$$

$$①^{-1} - ①^{-1} = 0, \qquad \frac{①^{-1}}{①^{-1}} = 1, \quad \frac{5 + ①^{-3.1}}{①^{-3.1}} = 5①^{3.1} + 1, \quad (①^{-1})^0 = 1,$$

$$① \cdot ①^{-1} = 1, \quad ① \cdot ①^{-3.1} = ①^{-2.1}, \quad \frac{①^{3.1} + 4①}{①} = ①^{2.1} + 4,$$

$$\frac{①^{3.1}}{①^{-3.1}} = ①^{6.2}, \quad (①^{3.1})^0 = 1, \quad ①^{3.1} \cdot ①^{-1} = ①^{2.1}, \quad ①^{3.1} \cdot ①^{-3.1} = 1.$$

It follows from (1) that $①^0 = 1$, therefore, a finite number a can be represented in the new numeral system simply as $a①^0 = a$, where the numeral a itself can be written down by any convenient numeral system used to express finite numbers. The simplest infinitesimal numbers are represented by numerals having only negative finite powers of ①, *e.g.* $50.1①^{-10.2} + 16.38①^{-20.3}$, also see examples above. Note that all infinitesimals are not equal to zero. In particular, $\frac{1}{①} > 0$ because it is a result of division of two positive numbers.

It should be mentioned that in certain cases ①-based numerals allow us to execute a finer analysis of infinite objects than traditional tools allow us to do. For instance, it becomes possible to measure certain infinite sets and to see, e.g., that the sets of even and odd numbers have ①/2 elements each. The set $\mathbb{Z}$ of integers has $2① + 1$ elements: ① positive elements, ① negative elements, and zero. Within the countable sets and sets having cardinality of the continuum, see [3, 15, 14], it becomes possible to distinguish infinite sets having different number of elements expressible in the numeral system using grossone and to see that, for instance,

$$\frac{①}{2} < ① - 1 < ① < ① + 1 < 2① + 1 < 2①^2 - 1 < 2①^2 < 2①^2 + 1 <$$

$$2①^2 + 2 < 2^{①} - 1 < 2^{①} < 2^{①} + 1 < 10^{①} < ①^{①} - 1 < ①^{①} < ①^{①} + 1.$$

It is important to stress that the new approach does not contradict Cantor's results. The situation is similar to what happens when one uses a microscope with two different lenses: the first of them is weak and allows one to see the object of the observation as two dots and another lens is stronger and allows the observer to see 10 dots instead of the former first dot and 32 dots instead of the former second dot. Both lenses give two correct answers having different accuracies. Analogously, both approaches, Cantor's and the new one, give correct answers but the accuracy of the answers is different. Cantor's tools say that the sets of even, odd, natural, and integer numbers have the same cardinality $\aleph_0$. This answer is correct with the precision that cardinal numbers have. However, the fact that they all have the same cardinality can be viewed also as the accuracy of the used instrument is too low to see that these sets have different numbers of elements. The new numeral system allows us to see these differences among sets having cardinality $\aleph_0$ and among sets having cardinality of the continuum, as well. See [3, 7, 15, 14, 18] for a detailed discussion including also the one-to-one correspondence issues.

We conclude this brief informal introduction by mentioning that properties of grossone are described by the Infinite Unit Axiom, see [15, 14, 18]. This new axiom is added to the axioms for real numbers. In the context of the present paper two issues postulated by the Axiom are important for us: (i) grossone is an infinite number; (ii) grossone is divisible by any finite integer. Note that grossone is not the only number enjoying the latter property. In fact, zero is also divisible by any finite integer.

2. Infinigons and infinigrids: classical approach

We remind the reader that we consider the Poincaré's disc model of the hyperbolic plane. We denote by D the once and for all fixed disc of the Euclidean plane which is the support of Poincaré's model. We denote by ∂D the circle which is the border of D. We remind the reader that the points of ∂D are called **points at infinity** and that they do not belong to the hyperbolic plane. The figures of the paper will take place in this frame. In our sequel, otherwise not mentioned, *line* means a line of the hyperbolic plane, most often an arc of a circle in the model. We refer the reader to [5], for instance, where other references are mentioned.

In this paper, we give a different proof from what was outlined in [4] although it is based on the same construction.

Fix two orthogonal diameters of D. One is called **horizontal** and the other one **vertical**. In order to define the infinigons, we consider the following sequence $\{x_n\}_{n\in\mathbb{Z}}$. Given two points x_n and x_{n+1} with $n \in \mathbb{N}$ and the angle $\alpha \in]0,\pi[$, we first construct the line β_{n+1} which passes through x_{n+1} and which makes an angle of $\alpha/2$ with the line $x_n x_{n+1}$. Define x_{n+2} to be the image of x_n by reflection in β_n. We repeat the process indefinitely, starting from $x_0 = \mathrm{O}$, where O is the centre of D, and from x_1 for the points with positive indices and starting from x_1 and x_0 for the points with negative indices.

Theorem 1. (Margenstern, see [4]) *For all α and x, the points x_n which are obtained by the construction above belong to a euclidean circle, call it Γ whose diameter is $\dfrac{x}{\cos(\frac{\alpha}{2})}$. Moreover, the curvilinear abscissa of the x_n's on Γ starting from x_0 toward x_1 are increasing. Γ is strictly inside D, is a horocycle or an equidistant curve if and only if $x < \cos(\dfrac{\alpha}{2})$, $x = \cos(\dfrac{\alpha}{2})$ or $x > \cos(\dfrac{\alpha}{2})$ respectively. If Γ is a horocycle, the x_n's converge to its unique point at infinity. If Γ is an equidistant curve, the x_n's for positive n converge to one point at infinity of Γ while the x_n's for negative n converge to the other point at infinity.*

The proof is illustrated by Figure 1.

Let μ_1 be the hyperbolic bisector of the segment $[x_0x_1]$. Then let β_1 be the hyperbolic line passing through x_1 which makes an angle of $\dfrac{\alpha}{2}$ with the line which supports x_0x_1. Note that we may consider that the first two points are on a diameter of D with x_0 at the centre of D. Depending on the possible intersection of μ_1 with β_1, we have three cases: either μ_1 and β_1 meet inside D, or they meet on ∂D or they do not meet at all.

First case: $\mu_1 \cap \beta_1 = A$, where A is a point of the hyperbolic plane, so that it is inside D, not on ∂D. Then, the triangle x_0x_1A is isosceles with x_0x_1 as its basis. Let x_{n+2}, for n a non-negative integer, be the reflection in the hyperbolic line Ax_{n+1} of x_n. If $x_nx_{n+1}A$ is isosceles with x_nx_{n+1} as its basis, the reflection $x_{n+1}x_{n+2}A$ is also isosceles. Clearly, all x_n's belong to a circle Γ around A which lies inside D: it is a circle of the hyperbolic plane as Γ has no intersection with ∂D.

Second case: $\mu_1 \cap \beta_1 = \mathbf{P}$, where $\mathbf{P}$ is on ∂D. This means that the lines μ_1 and β_1 are parallel. The triangle $x_0x_1\mathbf{P}$ is not an ordinary triangle. We call it an **ideal** triangle as it has one vertex on ∂D. It is also an isosceles triangle as $(x_1x_0, x_1\mathbf{P}) = (x_0x_1, x_0\mathbf{P})$ by parallelism: this comes from the fact that μ_1 is the bisector of $[x_0x_1]$. The same argument as previously holds with this time a horocycle Γ whose center is $\mathbf{P}$ and which passes through all x_n's too. In the Poincaré's disc model appears as a Euclidean circle which is tangent to ∂D at $\mathbf{P}$.

Third case: $\mu_1 \cap \beta_1 = \emptyset$. This time the line μ_1 and β_1 have a unique common perpendicular π. Consider the orthogonal projections y_n and y_{n+1} of x_n and x_{n+1} respectively on π. By the reflection in μ_1, $x_n y_n y_{n+1} x_{n+1}$ is a Saccheri quadrangle. Now, the reflection in $y_{n+1}x_{n+1}$ which keeps π globally invariant provides us with a new Saccheri quadrangle $x_{n+2}y_{n+2}y_{n+1}x_{n+1}$. Clearly the x_n's belong this time to an equidistant curve Γ whose direction is π. Angles $(x_{n+1}x_n, x_{n+1}x_{n+2})$ are all perpendicular to π. In the Poincaré's disc model it is a circle Γ which cuts ∂D at the two points of intersection of π with ∂D.

The computation of the Euclidean radius of Γ is left as an exercise. The corresponding easy computations can be found in [9].

In the case when $x = \cos(\frac{\alpha}{2})$, the convex hull of the x_n's, completed by the reflection in one of the β_n's is called an **infinigon**. It is not difficult to see that an infinigon does not occupy all the hyperbolic plane and that it tiles the plane if and only if $\frac{\alpha}{2} = \frac{2\pi}{k}$ for some k with $k \geq 3$.

However, there is another construction of the infinigons which will be of help for us. It was also indicated in [4]. It consists in considering a regular convex polygon P with $\frac{\pi}{2}$ as interior angle, by placing a vertex V at O, the centre of D, and one side abutting at V on a diameter of D, and the other side on a diameter which is orthogonal to the previous one. Next, with these conditions being fixed, we make the number of sides of P to grow. What happens? It happens the length of the side of P increases, but it reaches a limit, namely $\cos(\frac{\pi}{4})$ in this example. And so, the limit of the polygons is an infinigon as we have indicated. This can be generalized as follows:

Theorem 2. *Let $P_{p,q}$ denote the regular convex polygon such that one of its vertex is* O. *Let $\Gamma_{p,q}$ be the circumscribed circle of $P_{p,q}$. We may assume that the tangent at* O *to $\Gamma_{p,q}$ is horizontal. Let $h_{p,q}$ be the Euclidean distance of the hyperbolic centre of $\Gamma_{p,q}$ to* O *and let $e_{p,q}$ be the Euclidean distance of the Euclidean centre of $\Gamma_{p,q}$ to* O. *Then,*

$$h_{p,q} = \frac{\cos(\frac{\pi}{q} + \frac{\pi}{p})}{\sqrt{\cos^2 \frac{\pi}{q} - \sin^2 \frac{\pi}{p}}}. \tag{1}$$

and:

$$e_{p,q} = \frac{\cos(\frac{\pi}{q} + \frac{\pi}{p})\sqrt{\cos^2 \frac{\pi}{q} - \sin^2 \frac{\pi}{p}}}{\cos^2 \frac{\pi}{q} - \sin^2 \frac{\pi}{p} + \cos^2(\frac{\pi}{q} + \frac{\pi}{p})} \tag{2}$$

We have that for any q, $h_{p,q} \to 1$ as $p \to \infty$ and $e_{p,q} \to \frac{1}{2}$ as $p \to \infty$.

Let us consider this situation. Let $P_{p,q}$ be the regular convex polygon with p sides and an interior angle $\frac{2\pi}{q}$ so that q copies of $P_{p,q}$ can be put around a point A to cover a neighbourhood of A with no overlap. The vertices of $P_{p,q}$ can be obtained from As previously, we can put one vertex of the polygon at O and then, we proceed as in the proof of the theorem. As we know, the x_n's are on a circle $\Gamma_{p,q}$. In [8], we give the computation of the radius of $\Gamma_{p,q}$. Here we give a somewhat simplified version of this computation. The vertices of $P_{p,q}$ can be written as $r_{p,q}e^{i\vartheta}$ which we rewrite $re^{i\vartheta}$ as p and q are fixed in this part of the proof. We define ϑ by $\vartheta = -\dfrac{\pi}{p} + k\dfrac{2\pi}{p}$ with $k \in [0..p-1]$. Define A_k the vertex defined by k and consider A_0. The Euclidean support C of the segment A_0A_1 is the circle whose equation is $X^2 + Y^2 - 2\omega X + 1 = 0$, where $(\omega, 0)$ is the centre Ω of C. Let (x_0, y_0) be the coordinates of A_0. We write that the tangent of C at A_0 makes the angle $\dfrac{\pi}{q}$ with OA_0. As $\overrightarrow{\Omega A_0}$ has coordinates $(x_0 - \omega, y0)$, we can take $\overrightarrow{T}$ with coordinates $(y_0, \omega - x_0)$ for the tangent. As $\overrightarrow{\mathrm{O}A_0}.\overrightarrow{T} = \mathrm{O}A_0.|\overrightarrow{T}|.\cos\dfrac{\pi}{q}$. taking into account that $y_0 \neq 0$, easy computations show us that

$$\omega^2 = \frac{\cos^2\frac{\pi}{q}}{\cos^2\frac{\pi}{q} - \sin^2\frac{\pi}{p}} \qquad (a)$$

Now, we rewrite the fact that C passes through A_0 by $r^2 - 2\omega r\cos\dfrac{\pi}{p} + 1 = 0$, solving this equation in r and taking into account that $0 < r < 1$ should be satisfied, we get:

$$r = \omega\cos\frac{\pi}{p} - \sqrt{\omega^2\cos^2\frac{\pi}{p} - 1}. \qquad (b)$$

Easy but a bit tedious computations lead from (a) and (b) to (1). The computations which are omitted here can be found in [9].

In order to get the distance from the Euclidean centre of $\Gamma_{p,q}$ to O, let $h = h_{p,q}$ in order to simplify the notations and let s be the Euclidean length of the diameter of $\Gamma_{p,q}$. Of course, the required distance is $\dfrac{s}{2}$, so that we have to compute s. Now, let S be the point at distance s from O on a diameter of D which also passes through the centre of $\Gamma_{p,q}$ and let H denote the hyperbolic centre of $\Gamma_{p,q}$. We know that H is the hyperbolic mid-point of OS. Let C be the circle which passes through H, centred on the line OS and which is orthogonal to ∂D. Let $(\omega, 0)$ be the coordinates of C as OS can be taken as the x-axis. We have:

$$\omega(\omega - s) = (\omega - h)^2$$
$$h^2 - 2h\omega + 1 = 0$$

The first equation says that H is the mid-point of OS. The second one says that C passes through H. Cancelling ω^2 in the first equation and subtracting with the second one we get that $\omega s = 1$. Putting that in the second equation we get $h^2 s - 2h + s = 0$ from which we derive $s = \dfrac{2h}{1 + h^2}$. Using (1) in this latter expression, we get (2) after straightforward computations.

The conclusion of the Theorem on the limits of $h_{p,q}$ and $e_{p,q}$ are straightforward. □

We can see that the conclusion of Theorem 2 confirms the statement of Theorem 1.

3. Infinigons, infinigrids and grossone

What can be said about these construction in the light of the new numeral system introduced by Yaroslav Sergeyev, see [11, 12, 13, 16, 17]? Here, we mainly look at the cases when $x = \cos(\frac{\alpha}{2})$ and when $\alpha = \frac{2\pi}{q}$ for some positive integer q.

Let us consider the construction involved in Theorem 1. We can see that we must replace the vague notion of ∞ with the more precise indication on the infinite number of x_n's we consider. In this section and in the next one, we shall use λ and μ to denote such infinite numbers which can be replaced by ①, or $\frac{①}{2}$, or any expression involving ①, as those indicated in Section 1. The conclusion we must reach is that if λ is a positive infinite integer, x_λ never reaches **P**. Indeed, by construction, x_λ is in the hyperbolic plane, *i.e.* inside D, so that if $x_{\lambda+1}$ can be defined, $x_{\lambda+1}$ is also in the hyperbolic plane and it cannot be **P**. However, **P** itself can be defined, at least as the intersection of β_1 and μ_1 at infinity or, which is equivalent, by saying that β_1 and μ_1 are parallel. Accordingly, we can say that in some sense, the infinigon can also be defined, but it is an **ideal** object in this sense that it is essentially incomplete: we cannot tell the number of its sides.

If we look at the construction which is considered in Theorem 2, we have a completely different landscape. This time, if λ is a positive infinite integer, we can define a regular convex polygon P with λ sides. The computations performed in Section 2 for Theorem 2 gives us a precise description of this object. Consider that q is fixed, where q is a positive finite integer. Then the vertices of P are on a circle Γ of the hyperbolic plane which is not a horocycle: this circle is completely in the hyperbolic plane. However, its diameter is infinite as its representation in the hyperbolic plane is infinitesimally close to 1.

Replacing p by λ in (2), we can see that the diameter of Γ is

$$\frac{2\cos(\frac{\pi}{q}+\frac{\pi}{\lambda})\sqrt{\cos^2\frac{\pi}{q}-\sin^2\frac{\pi}{\lambda}}}{\cos^2\frac{\pi}{q}-\sin^2\frac{\pi}{\lambda}+\cos^2(\frac{\pi}{q}+\frac{\pi}{\lambda})}.$$

Let us put $a = \cos(\frac{\pi}{q}+\frac{\pi}{\lambda})$ and $b = \cos^2\frac{\pi}{q}-\sin^2\frac{\pi}{\lambda}$. Then, $d = \frac{2a\sqrt{b}}{a^2+b}$. Is it true that $d < 1$? Indeed, $d < 1$ if and only if $2a\sqrt{b} < a^2+b$ *i.e.* if and only if $4a^2b < a^4+b^2+2a^2b$ which is equivalent to $a^4+b^2-2a^2b > 0$. Easy computations just involving trigonometric formulas, see [9], allow us to prove that $a^2 \neq b$ also using $0 < \frac{\pi}{q}+\frac{\pi}{\lambda} < \frac{\pi}{2}$ which simply means that the isosceles triangle which constitute P are true triangles in the hyperbolic plane.

The computations prove that:

$$b - a^2 = 2\sin\frac{\pi}{q}\sin\frac{\pi}{\lambda}\cos(\frac{\pi}{q}+\frac{\pi}{\lambda}) \qquad (3)$$

which is our initial claim. Now, consequently, we obtain that that $d < 1$. This shows that Γ and P both remain in the hyperbolic plane.

The just provided computation can be made more precise: we know that $(a^2+b)^2 - (2a\sqrt{b})^2 = (a^2-b)^2$ so that from (3) as $d = 2h_{p,\lambda}$, we can see that $1 - d^2 = (\dfrac{b-a^2}{a^2+b})^2$. When λ is an infinite positive integer, the order of a^2+b is $2\cos^2\dfrac{\pi}{q}$ and that of $b-a^2$ is $2\sin\dfrac{\pi}{q}\cos\dfrac{\pi}{q}$, so that

$$d^2 \approx 1 - \frac{\sin^2\frac{\pi}{q}}{\cos^2\frac{\pi}{q}}\frac{\pi^2}{\lambda^2} \tag{4}$$

.

This inequality proves that Γ is still in the hyperbolic plane. Call **infinigon** this polygon with infinitely many sides, exactly with λ of them. It is plain that such an infinigon tiles the hyperbolic plane.

In fact, from (1) and from the fact that the hyperbolic distance δ corresponding to the Euclidean distance d from O to S is given by Lobachevsky's formula: $\delta = \ln\left|\dfrac{1+d}{1-d}\right|$. We obtain in this way that $\delta = 2\ln\Big(\dfrac{a+\sqrt{b}}{\sqrt{b}-a}\Big)$ so that in the end $\delta = 2\ln(a+\sqrt{b}) - 2\ln(\sqrt{b}-a)$. Now,:

$$\ln(a+\sqrt{b}) = 2\ln\Big(\cos(\frac{\pi}{q}+\frac{\pi}{\lambda}) + \sqrt{\cos^2\frac{\pi}{q} - \sin^2\frac{\pi}{\lambda}}\Big)$$

which is equal to $2\ln\Big(\cos(\dfrac{\pi}{q}+\dfrac{A}{\lambda})\Big)$, where A is a function of q and λ bounded by a finite positive number. Going on trigonometric computations given in [9] and taking into account that $\tan\dfrac{\pi}{\lambda}$ differs from $\dfrac{\pi}{\lambda}$ by a higher order infinitesimal, and that $\dfrac{1}{\lambda}$ is an infinitesimal which is infinitely bigger than $\dfrac{1}{\lambda^2}$, we get that

$$\ln(\sqrt{b}-a) \approx 2\ln\Big(\frac{\pi}{\lambda}\sin\frac{\pi}{q}\Big) = -2\ln\lambda + 2\ln\pi + 2\ln\sin\frac{\pi}{q}.$$

As $\delta = 2\ln(a+\sqrt{b}) - 2\ln(\sqrt{b}-a)$, we eventually get that $\delta \approx 2\ln\lambda$, so that δ is an infinite number.

Now, from (1) and (2) we have something more: if we replace q by a positive infinite integer μ, the formulas are still valid as well as the computation leading to formula (3). Note that μ may be different from λ: μ can even be chosen independently from the value of λ. In this case we can replace the estimation given in (4) by the following one:

$$d^2 \approx 1 - \frac{\pi^4}{\lambda^2\mu^2} \text{ and } \delta \approx \frac{\pi^2}{\lambda\mu} \approx -2\ln\lambda - 2\ln\mu. \tag{5}$$

Again we call **infinigon** the polygon obtained in this case as its diameter is actually infinite and as the number of its sides is also defined by an infinite integer. However, in order to distinguish between these two kinds of infinigons, call **infinigon of first order**, for short **first order infinigon**, those defined by $P_{q,\lambda}$ where λ is a positive infinite integer and q a positive finite integer with

$q \geq 3$. We call **infinigon of second order**, for short **second order infinigon**, those defined by $P_{\mu,\lambda}$ where λ and μ are both positive infinite integers, possibly different integers. It is plain that both $P_{q,\lambda}$ and $P_{\mu,\lambda}$ tile the plane by the standard process: we take P_0 a copy of $P_{q,\lambda}$ and then we replicate it by reflection in its sides and, recursively, by reflections of the images in their sides. The same process can be applied to copies of $P_{\mu,\lambda}$ where both μ and λ are infinite numbers.

We have still one point to investigate.

When we say that the infinigons of first order tile the plane by the above process, we say *recursively* which is in fact a vague term. In the traditional meaning, this means *endlessly*. As there is no more precise notion than the cardinals for estimating infinite numbers in traditional mathematics, here we have to make things more clear. When we say *recursively* we have to mention to which depth we go on the recursive process. Controlling recursion up to a fixed depth in advance is a standard feature in the implementation of certain programming languages. This does not prevent more theoretic oriented languages to allow depths which are only limited by the resources of the machine on which the program runs. Here, we adopt the same spirit: when we use the word *recursively*, it is possible to not indicate to which depth, but for a precise study of the process, it is better to indicate to which depth we allow to proceed. Let ν be the depth of recursion. It is plain that from this definition, after a few steps of iteration of the process, we may obtain a copy which overlaps an already existing copy. Of course, a copy is considered to be reached by the k^{th} recursive call if it has not been produced by a previous call.

A way to detect the set of copies obtained up to the depth n has been indicated in [4]. It consists in building a tree which is in bijection with the tiling. However, as [4] was written with a more traditional look at infinity, we have to revisit this construction.

Say that the centre of an infinigon is the centre of its circumscribing circle. We fix two contiguous sides of P_0, a fixed copy of $P_{q,\lambda}$, say s_0 and s_1 and let V_0 be their common vertex. We have two cases depending on whether q is odd or even.

First assume that q is even, say $q = 2h$. This is the easy case. Consider the ray ℓ which is issued from V_0 and which supports s_0. We may consider that s_0 lies on the left hand side of V_0. Let V_0^1 be the other end of s_0. From V_0^1, out of s_0, ℓ is the support of a side shared by two copies of P_0 which share V_0^1 with P_0. It is plain that this process can be continued. The same can be performed with m, the ray issued from V_0 which supports s_1. Let $\mathcal{S}$ be the angular sector defined by the angle (ℓ, m). We construct a tree whose root is attached to P_0. Each node of the tree is attached to a copy of P_0 inside $\mathcal{S}$. A node ν of the tree is the son of a node π only if the copy attached to ν and that attached to π share a common side. To precisely define the notion of childhood, we start from the root. By definition, its sons are the reflections of P_0 in its sides which are still inside $\mathcal{S}$. Now, consider σ_k, $k \in [1..\lambda-2]$, the other sides of P_0, starting from V_0^1 and counter-clockwise turning around P_0. Let P_k be the reflection of P_0 in σ_k. Let V_0^k be the vertex shared by σ_{k-1} and σ_k with $\sigma_0 = s_0$. Then, around V_0^k, there are q copies of P_0, P_0 being taken into account. If we recursively repeat the process of copying starting from the P_k's, there will be overlapping: starting from P_1 and looking at its sides on the right-hand side, after $q-2$ iterations, we get P_2. Accordingly, we have to give rules in order to avoid overlapping. To this purpose, we shall say that the P_k's we have just defined with $k \in [1..\lambda - 2]$ are the **main sons**. Each main son has j **brothers**, j being $q-3$ or $q-4$, depending on a circumstance to which we turn now.

Consider σ_k, a side of P_0 in the angle (ℓ, m) with $k > 1$ and $k < \lambda-2$. The vertices of σ_k are V_0^k and V_0^{k-1}. In order to delimit regions which do not overlap but completely cover the complement of P_0 in the angle (ℓ, m), we continue σ_k a,d σ_{k+1} both to the left. Now, it is easy to see that around V_0^k and outside P_0, we have h copies of the angle (ℓ, m), so that outside P_k and around V_0^k there are $h-1$ copies of (ℓ, m). On the other side, there are $h-2$ copies of (ℓ, m) as from the continuation of σ_{k+1} we have to take into account the angle which is in P_0 and that which is in P_k. So that for $1 < k < \lambda-2$, in each region (σ_k, σ_{k+1}) and outside P_k $q-3$ copies of (ℓ, m). We remain with the examination of $k = 1$ and $k = \lambda-2$. When $k = \lambda-2$, s_1 plays the role of σ_{k+1}, so that in this case, the number of angles left in the region $(\sigma_{\lambda-2}, s_1)$, is also $q-3$. When $k = 1$, The region is (s_0, σ_1) which is smaller than a region (σ_k, σ_{k+1}) with $k > 1$. The region is smaller than π by an angle which is equal to (ℓ, m), so that this time we have $h-2$ copies of (ℓ, m) close to V_0 1 and $h-2$ of them to close to V_0^2, so that we get $q-4$ copies of (ℓ, m). Accordingly, in this case, we can see that it is possible to split (ℓ, m) into $1 + \lambda-2$ copies of P_0 and $(\lambda-3)(q-3) + q-4$ copies of $(\ell, m)_2$ where $(\ell, m)_2$ has the same angle as (ℓ, m) but with a depth reduced by 2: if P_0 has recursion depth κ, with κ finite or infinite integer, P_k with $k \in [1..\lambda-2]$ has depth $\kappa+1$, so that any infinigon in an $(\ell, m)_2$ has depth at least $\kappa+2$. Note that $h-1$ steps of recursion are needed in order the vertices of P_0 should be completely covered.

Now, consider the case when q is odd, we shall write $q = 2h+1$.

This time, the regions have to be changed. Consider the previous setting with the same notations. If we continue the side σ_k, it is not a side of an adjacent infinigon: the continuation is a bisector of an interior angle of an infinigon which shares V_0^k with σ_k. Now, to consider half-infinigons is possible only if λ is even. If λ too is odd, then we get a more complex situation.

However, using a trick we explained in [6], we can handle the situation no matter which the parity of λ is. The idea consists in replacing the regions we considered when q is even by new regions to which we now turn. Consider the mid-pointM_0 of the side s_0 of P_0. Let N_0 be the mid-point of the side τ_0 which abuts s_0 at V_0^1 and which makes an angle $\vartheta = h\dfrac{\pi}{q}$ with s_0 by going outside of P_0. Let W_0 be the other end of τ_0. Let R_0 be the mid-point of the side ω_0 which abuts τ_0 at W_0 and which makes an angle $\varphi_0 = h\dfrac{\pi}{q}$ with τ_0, the angles ϑ_0 and φ_0 being on different sides of τ_0. From the construction, the isosceles triangles $M_0V_0^1N_0$ and $N_0W_0R_0$ are equal so that the points M_0, N_0 and R_0 lie on a same ray u issued from M_0 whose supporting line is called a **h-mid-point line** in [6]. A similar h-mid-point line v can be drawn from the mid-point of s_1 which is symmetric to u with respect to the bisector β of the angle (ℓ, m). Clearly, u and v meet onβ inside P_0. Such an (u, v) is called an **angular sector** (u, v) and we may distinguish between copies of it depending on the recursion depth of the copy. In such an angular sector (u, v), outside P_0, we take all infinigons such that all mid-points of their sides lie inside the angle (u, v) or, possibly, on u or on v.

Using the construction of the rays u and v, we can define a process which is similar to the one we defined for the case when q is even. This time, for each σ_k with $1 < k < \lambda{-}2$, we consider the rays issued from the mid-points of σ_{k-1} and σ_{k+1} supported by h-mid-point lines with respect to σ_{k-1} on one side and with respect to σ_{k+1} on the other side. This define new regions which we call **strips**. All these strips are equal and, indeed, the equality also holds for $k = 1$ and $k = \lambda{-}1$. Besides P_k, each strip contains $q{-}4$ copies of an angular sectors (u, v) with a smaller depth, smaller by 2.

Note that the sides of infinigons which cross u and$v$ can be used to define the depth of the recursion. Consider an angular sector (u, v) as defined above. Afters_0, denote by s_i the sides of infinigons which cross the h-mid-point line which supports u. We have that s_{2i} goes inside the sector while s_{2i+1} goes outside. The infinigon whose side is s_{j+1} is reached from that whose side is s_j after $h{-}1$ reflections. It is more natural to count the recursion depth in this way so that after one recursion step, the vertices of the previous generation of infinigons are completely covered by the new one. We shall now take this definition of the recursion depth which we also call generation.

Accordingly, if N_{k+1} is the number of infinigons generated at the $k{+}1^{\text{th}}$ generation, then, from what we have proved we can see that $N_{k+1} = (\lambda{-}2)(q{-}4)N_k$ when q is odd and $N_{k+1} = ((\lambda{-}2)(q{-}3){-}1)N_k$ when q is even as:

$$(\lambda{-}3)(q{-}3){+}q{-}4 = (\lambda{-}2)(q{-}3){-}1.$$

We can sum up our study by the following result:

Theorem 3. *There are two kinds of regular convex infinigons in the hyperbolic plane: those which have λ sides and $\dfrac{2\pi}{q}$ as their interior angle, where λ is an infinite integer and q is a finite one, and those which have λ sides and $\dfrac{2\pi}{\mu}$ as their interior angle, where both λ and μ are infinite integers. The first kind of infinigons are said of the* **first order** *and the second kind are said of* **second order**. *Both first order and second order infinigons lie completely in the hyperbolic plane with no point at infinity: in both cases, there is a hyperbolic circle, whose radius is infinite, which circumscribes all the vertices. The radius $\rho_{\lambda,p}$, $\rho_{\lambda,\mu}$ of these circles, when passing through* O, *the centre of the Poincaré's disc is given by the following formulas:*

$$\rho_{\lambda,q} = \frac{\cos(\frac{\pi}{q} + \frac{\pi}{\lambda})}{\sqrt{\cos^2\frac{\pi}{q} - \sin^2\frac{\pi}{\lambda}}}, \textit{ and } \rho_{\lambda,\mu} = \frac{\cos(\frac{\pi}{\mu} + \frac{\pi}{\lambda})}{\sqrt{\cos^2\frac{\pi}{\mu} - \sin^2\frac{\pi}{\lambda}}}.$$

where λ and μ are infinite integers, the left-, right-hand side formula applying to first, second order infinigons respectively. Both kinds of infinigons tile the *hyperbolic plane, giving rise to two kinds of infinite families of tilings:* $\mathcal{T}_{\lambda,q,\nu}$ *and* $\mathcal{T}_{\lambda,\mu,\nu}$*, where* ν *is an infinite positive integer, indicating the depth of the recursion used to define the tiling. For first order infinigons, the number of tiles in* $\mathcal{T}_{\lambda,q,\nu}$ *is* $((\lambda-2)(q-4))^\nu$ *when* q *is odd, and it is* $((\lambda-2)(q-3)-1)^\nu$ *when* q *is even. For second order infinigons, the number of tiles in* $\mathcal{T}_{\lambda,\mu,\nu}$ *is* $(\lambda-2)(\mu-4))^\nu$ *when* μ *is odd and it is* $((\lambda-2)(\mu-3)-1)^\nu$ *when* μ *is even.*

Note that as $q-4$ is $2h-3$ when $q = 2h+1$ and $q-3$ is $2h-3$ too when $q = 2h$.

Acknowledgment

The author is very much in debt to Yaroslav Sergeyev for his interest to this work. He also wishes to thank the referees for their remarks allowing him to improve the presentation of my results.

References

[1] H.S.M. Coxeter, Regular honeycombs in hyperbolic space, *International Congress of Mathematicians*, (1954), **3**, 155–169.

[2] P. Gordon. Numerical cognition without words: Evidence from Amazonia, *Science*, 306(15 October):496–499, 2004.

[3] G. Lolli, Infinitesimals and infinites in the history of Mathematics: A brief survey, *Applied Mathematics and Computation*, **218**(16), (2012), 7979-7988.

[4] M. Margenstern, On the Infinigons of the Hyperbolic Plane, A combinatorial approach, *Fundamenta Informaticae*, **56**(3), (2003), 255-272.

[5] M. Margenstern, Cellular Automata in Hyperbolic Spaces, Volume 1, Theory, *OCP*, Philadelphia, (2007), 422p.

[6] M. Margenstern, About a new splitting for the algorithmic study of the tilings $\{p, q\}$ of the hyperbolic plane when q is odd, *arXiv*:0911.4040v2, (2009), 20pp.

[7] M. Margenstern, Using grossone to count the number of elements of infinite sets and the connection with bijections. *p-Adic Numbers, Ultrametric Analysis and Applications*, **3**(3), (2011), 196–204.

[8] M. Margenstern, *Small Universal Cellular Automata in Hyperbolic Spaces, A Collection of Jewels*, Springer, Emergence, Complexity and Computation Series, Ivan Zelinka, Guanrong Chen, Andrew Adamatzky Editors, (2013), 320pp.

[9] M. Margenstern, Infinigons of the hyperbolic plane and grossone, *arXiv*:1502.07990v2, (2015), 13pp.

[10] B.A. Rozenfeld, Neevklidovy prostranstva, Nauka, Moscow, 1969, 547p.

[11] Ya. D. Sergeyev, *Arithmetic of Infinity*, Edizioni Orizzonti Meridionali, CS, (2003).

[12] Ya. D. Sergeyev, Blinking fractals and their quantitative analysis using infinite and infinitesimal numbers, *Chaos, Solitons & Fractals*, **33**(1), (2007), 50-75.

[13] Ya. D. Sergeyev, A new applied approach for executing computations with infinite and infinitesimal quantities, *Informatica*, **19**(4), (2008), 567-596.

[14] Ya.D. Sergeyev. Lagrange Lecture: Methodology of numerical computations with infinities and infinitesimals. *Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino*, **68**(2), (2010), 95–113.

[15] Ya.D. Sergeyev. Counting systems and the First Hilbert problem. *Nonlinear Analysis Series A: Theory, Methods & Applications*, **72**(3-4), (2010), 1701–1708.

[16] Ya.D. Sergeyev, Using blinking fractals for mathematical modelling of processes of growth in biological systems, *Informatica*, **22**(4), (2011), 559-576.

[17] Ya.D. Sergeyev, Solving ordinary differential equations by working with infinitesimals numerically on the Infinity Computer, *Applied Mathematics and Computation*, **219**(22), (2013), 10668-10681.

[18] Ya.D. Sergeyev. Numerical computations with infinite and infinitesimal numbers: Theory and applications, In Sorokin A. and Pardalos P.M., editors, *Dynamics of Information Systems: Algorithmic Approaches*, (2013), pages 1–66. Springer, New York.

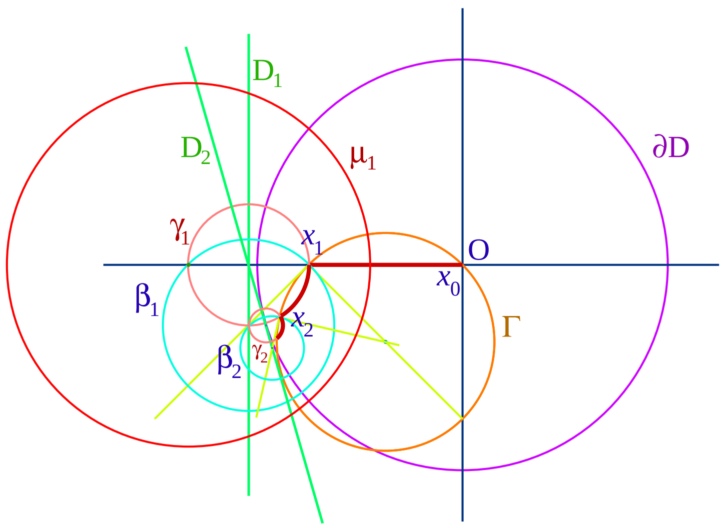

Figure 1. *The construction described in the proof of Theorem* 1.

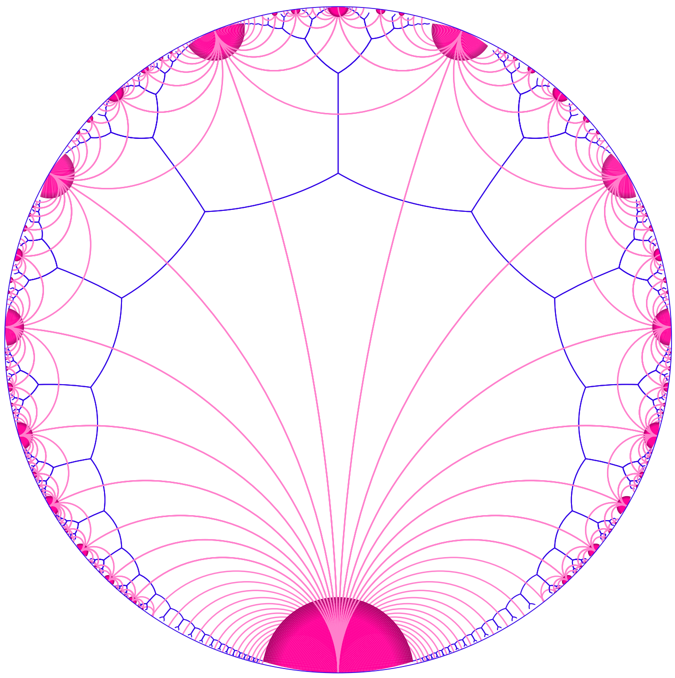

Figure 2. *An illustration for the first order infinigons. In this picture,* $q = 3$.