Analysis on non-Archimedean Field Extensions of the Real Numbers and Applications

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In this talk, we will give an overview of our work on non-Archimedean ordered field extensions of the real numbers that are real closed and complete in the order topology. The smallest such field, the Levi-Civita field $\mathcal{R}$ [1], is small enough to allow for the calculus on the field to be implemented on a computer and used in applications such as the fast and accurate computation of the derivatives of real functions up to very high orders [2].

We will summarize the convergence and analytical properties of power series, showing that they have the same smoothness behavior as real power series; and we will present a Lebesgue-like measure and integration theory on $\mathcal{R}$. Moreover, based on continuity and differentiability concepts that are stronger than the topological ones, solutions to one-dimensional and multi-dimensional optimization problems will be outlined.

A natural inner product can be defined on c_0, the space of null sequences of elements of $\mathcal{C} := \mathcal{R} + i\mathcal{R}$, which induces the sup-norm of c_0. Unlike classical Hilbert spaces, c_0 is not orthomodular with respect to this inner product, so we characterize those closed subspaces of c_0 with an orthonormal complement. We will present characterizations of normal projections, adjoint and self-adjoint operators, and compact operators on c_0 [3]. Then we will study in details the properties of positive operators on c_0.

References

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